JOURNAL OF APPROXIMATION THEORY 67, 284–302 (1991)

# A Global Inverse Theorem on Simultaneous Approximation by Bernstein–Durrmeyer Operators

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*Communicated by Zeev Ditzian*

Received January 30, 1990; revised December 1, 1990

DEDICATED TO PROFESSOR DR. W. HAUSSMANN ON THE OCCASION OF HIS 50TH BIRTHDAY

We prove a global inverse result for simultaneous approximation by modified Bernstein operators as introduced by Durrmeyer in 1967. The main result of this note supplements and extends an earlier direct theorem of Heilmann and Müller and is given in terms of the so-called Ditzian–Totik modulus of second order.
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## 1. INTRODUCTION

The classical Bernstein operators are of the form

$$B_n(f; x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_{n,k}(x),$$

where

$$p_{n,k}(x) = \frac{n!}{k!(n-k)!} x^k (1-x)^{n-k}, \qquad 0 \leqslant k \leqslant n.$$



0021-9045/91 $3.00
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There are two modifications of the Bernstein polynomials for the approximation of $L_p$ functions, $1 \leqslant p < \infty$, which have attracted particular interest over the recent years. The first is given by Kantorovich operators $B_n^*$ which are obtained if one replaces $f(k/n)$ by

$$(n+1) \int_{k/(n+1)}^{(k+1)/(n+1)} f(t)\, dt.$$

See [6] and the references cited there for details.

The other modification is an operator sequence introduced by Durrmeyer [7] and, independently, by Lupaş [11, p. 68]. Here, $f(k/n)$ is replaced by

$$(n+1) \int_0^1 p_{n,k}(t)\, f(t)\, dt,$$

so that one arrives at

$$M_n(f; x) = (n+1) \sum_{k=0}^{n} p_{n,k}(x) \int_0^1 p_{n,k}(t)\, f(t)\, dt.$$

The $M_n$ were studied by Derriennic [3, 4] and several other authors. It turned out that the approximation properties of both $B_n^*$ and $M_n$ are somewhat similar.

Writing $L_n$ for either $B_n^*$ or $M_n$, the following statements hold:

THEOREM A (See [5, 17]). *Let* $1 \leqslant p < \infty$. *Then, for* $0 < \alpha < 2$,

$$\|L_n f - f\|_p = O(n^{-\alpha/2})$$

*if and only if*

$$\omega_\varphi^2(f, t)_p = O(t^\alpha).$$

*Here*

$$\omega_\varphi^k(f, t)_p = \sup_{0 < h \leqslant t} \|\Delta_{h\varphi}^k\|_p, \qquad f \in L_p(I),\ \varphi(x) = (x(1-x))^{1/2},$$

*with*

$$\Delta_t f(x) = f\left(x + \frac{t}{2}\right) - f\left(x - \frac{t}{2}\right), \qquad \Delta_t^k f(x) = \Delta_t \Delta_t^{k-1} f(x),$$

*is the so-called Ditzian–Totik modulus of smoothness.*

For the saturation case one has

THEOREM B (See [8, 12–14, 16]).

$$\|L_n f - f\|_p = O(n^{-1})$$

*if and only if*

(1) *for* $1 < p < \infty$

$$\omega_\varphi^2(f, t)_p = O(t^2),$$

(2) *for* $p = 1$

$$f(x) = K + \int_y^x \frac{h(t)}{t(1-t)}\, dt \qquad \text{a.e. on } I,$$

*with* $y \in (0, 1)$ *and* $h(0) = h(1) = 0$, $h \in BV(I)$.

It would be desirable to have a more uniform description of the non-optimal and the saturation cases. However, as was shown by Totik [15], the condition given in Theorem B for $p = 1$ and

$$\omega_\varphi^2(f, t)_1 = O(t^2)$$

are not equivalent.

It is the aim of the present paper to show that one gets a more elegant characterization (at least for the operators $M_n$) if one considers simultaneous approximation. The direct part of the result below was, for the most part, established in a recent paper by Heilmann and Müller [9]. They stated, however, that they were unable to prove the inverse theorem for non-weighted global approximation. This will be done below. To be more specific, we shall show, among other results, that, for $1 \leqslant s$ fixed, one has for $1 \leqslant p < \infty$ the following equivalence (see the Theorem below):

For $0 < \alpha \leqslant 2$, $0 \leqslant \beta < \infty$,

$$\|(M_n f - f)^{(s)}\|_p \leqslant C\{n^{-\alpha/2}(\log n)^\beta\} \Leftrightarrow \omega_\varphi^2(f^{(s)}, t)_p = Ct^\alpha\, |\log t|^\beta.$$

Thus, not only is it true that there is a more elegant result for simultaneous approximation, but we can also characterize more classes of functions by the result of this note.

## 2. Auxiliary Results

If $f \in L_p^s(I)$, $1 \leqslant p \leqslant \infty$, $n > s$, then it was shown by Derriennic [4, p. 334] that

$$(M_n f)^{(s)}(x) = (n+1)\,\alpha(n,s) \sum_{k=0}^{n-s} p_{n-s,k}(x) \int_0^1 p_{n+s,k+s}(t)\, f^{(s)}(t)\, dt$$

with

$$\alpha(n,s) = \frac{(n!)^2}{(n-s)!\,(n+s)!}.$$

Heilmann and Müller [9] introduced the auxiliary operators

$$(M_{n,s} h)(x) = (n+1)\,\alpha(n,s) \sum_{k=0}^{n-s} p_{n-s,k}(x) \int_0^1 p_{n+s,k+s}(t)\, h(t)\, dt, \qquad h \in L_p(I).$$

They used the equality

$$(M_n f)^{(s)} = M_{n,s}(f^{(s)}), \qquad f \in L_p^s(I),$$

and mentioned that for $h \in L_p(I)$, $n > s$,

$$\|M_{n,s} h\|_p \leqslant C\, \|h\|_p$$

with a constant $C$ independent of $n$ and $p$.

While these results will be useful for us in the sequel, for convenience we summarize some further results which can be found in [5, 10] or can be obtained using similar methods.

Note first that, for $\varphi(x) = (x(1-x))^{1/2}$, the following relationships hold true:

$$\begin{aligned}
&\varphi^4(x)\, p_{n-s-2,k}(x)\, p_{n+s+2,k+s+2}(t) \\
&\quad \sim \varphi^2(x)\, \varphi^2(t)\, p_{n-s,k+1}(x)\, p_{n+s,s+k+1}(t) \\
&\quad \sim \varphi^4(t)\, p_{n-s+2,k+2}(x)\, p_{n+s-2,k+s}(t),
\end{aligned} \tag{1}$$

$$p'_{n,k}(x) = \frac{(k-nx)}{\varphi^2(x)}\, p_{n,k}(x) = n(p_{n-1,k-1}(x) - p_{n-1,k}(x)), \tag{2}$$

and, for suitably chosen $F$,

$$\begin{aligned}
&\left| \varphi^2(x) \frac{d^{s+2}}{dx^{s+2}} M_n(F; x) \right| \\
&\qquad \leqslant Cn \sum_{k=0}^{n-s-2} p_{n-s,k+1}(x) \int_0^1 p_{n+s,k+s+1}(t)\, \varphi^2(t)\, |F^{(s+2)}(t)|\, dt.
\end{aligned} \tag{3}$$

Let $a_k = \int_0^1 p_{n+s,k+s}(t)\, F^{(s)}(t)\, dt$ and $\Delta a_k = a_{k+1} - a_k$. Then, for $n \geqslant s+2$,

$$\frac{d^{s+2}}{dx^{s+2}} M_n(F; x) \sim n^3 \sum_{k=0}^{n-s-2} p_{n-s-2,k}(x)\, \Delta^2 a_k. \tag{4}$$

Here $A \sim B$ means that there exists a constant $C > 0$, such that

$$C^{-1} |B| \leqslant |A| \leqslant C |B|.$$

Let $N_i \in N$, $|N_i| < C$, $i = 1, ..., 5$, and $m \in N_0$ be fixed. Then

$$\left| n \sum_{k=0}^{n+N_1} p_{n+N_2,k+N_3}(x) \int_0^1 (x-t)^m\, p_{n+N_4,k+N_5}(t)\, dt \right| \leqslant C \left( \frac{\varphi(x)}{\sqrt{n}} + \frac{1}{n} \right)^m. \tag{5}$$

The following lemma can be found in [18]; its proof can be carried out similarly as in [1].

LEMMA 1. *Let $U_1(x)$, $U_2(x)$ be non-negative increasing functions, $r > 0$, $C > 1$. If for all $0 < t$, $h \leqslant 1$ one has*

$$U_1(t) \leqslant C \left\{ U_2(h) + \left( \frac{t}{h} \right)^r U_1(h) \right\},$$

*then*

$$U_1(h) \leqslant A \left\{ h^{r-1/2} \int_h^1 \frac{U_2(t)}{t^{r+1-1/2}}\, dt + h^{r-1/2} \right\},$$

*where $A$ depends on $C$, $U_1(1)$ and $U_2(1)$.*

Putting $E_n = [1/(n+\varepsilon), 1 - 1/(n+\varepsilon)]$ for some fixed $\varepsilon > 0$ ($\varepsilon$ not being the same at each occurrence), we have

LEMMA 2. *For $x \in E_n$, let $\varphi_n(x) = \varphi(x)/\sqrt{n}$. Then*

$$|p_{n,k}^{(j)}(x)| \leqslant C \left\{ \varphi_n^{-j}(x) \sum_{i=0}^{j} \frac{|k/n - x|^i}{\varphi_n^i(x)}\, p_{n,k}(x) \right\}, \qquad j = 1, 2, ...$$

*Proof.* For $j = 1$, (2) implies the above. If it is true for $j = j_0$, then for $j = j_0 + 1$. Since

$$\left| \frac{d^\mu}{dx^\mu} \varphi_n^{-2}(x) \left( \frac{k}{n} - x \right) \right| \leqslant C \left\{ \varphi_n^{-\mu-2} \left| \frac{k}{n} - x \right| + \varphi_n^{-\mu-1}(x) \right\},$$

we have, using (2),

$$\begin{aligned} |p_{n,k}^{(j_0+1)}(x)| &= \left| \frac{d^{j_0}}{dx^{j_0}} \varphi_n^{-2}(x) \left(\frac{k}{n}-x\right) p_{n,k}(x) \right| \\ &\leqslant C \left\{ \sum_{\mu=0}^{j_0} \varphi_n^{-j_0-2}(x) \sum_{i=0}^{j_0-\mu} \frac{|k/n-x|^{i+1}}{\varphi_n^i(x)} p_{n,k}(x) \right. \\ &\quad \left. + \varphi_n^{-j_0-1}(x) \sum_{\mu=0}^{j_0} \sum_{i=0}^{j_0-\mu} \frac{|k/n-x|^i}{\varphi_n^i(x)} p_{n,k}(x) \right\} \\ &\leqslant C\varphi_n^{-j_0-1}(x) \sum_{i=0}^{j_0+1} \frac{|k/n-x|^i}{\varphi_n^i(x)} p_{n,k}(x). \end{aligned}$$

That is what we want. ∎

Lemma 3. *For $1 \leqslant p \leqslant \infty$, the following inequalities holds*:

$$\|\varphi^{2i} M_{n,s}^{(2i)}(f)\|_p \leqslant C \, \|\varphi^{2i} f^{(2i)}\|_p, \qquad \varphi^{2i} f^{(2i)} \in L_p(I), \tag{6}$$

$$\|\varphi^{2i} M_{n,s}^{(2i)}(f)\|_p \leqslant C n^i \, \|f\|_p, \qquad f \in L_p(I), \tag{7}$$

*where* $i = 1, 2$.

$$\|M'_{n,s}(f)\|_p \leqslant Cn \, \|f\|_p, \qquad f \in L_p(I). \tag{8}$$

*Proof.* Because the remaining inequalities can be shown in the same way, we only prove (6) and (7) for $i=1$. We first represent $M''_{n,s}(f)$ using $M_n$. Let $F$ be such that $F^{(s)} = f$. Then

$$M_{n,s}(f) = (M_n F)^{(s)},$$

or

$$M''_{n,s}(f) = (M_n F)^{(s+2)}.$$

Using (3), one gets

$$\begin{aligned} \|\varphi^2 M''_{n,s}(f)\|_\infty &\leqslant Cn \left\| \sum_{k=0}^{n-s-2} p_{n-s,k+1}(x) \frac{1}{n+s+1} \right\|_\infty \|\varphi^2 f''\|_\infty \\ &\leqslant C \, \|\varphi^2 f''\|_\infty. \end{aligned}$$

Furthermore, for $p = 1$,

$$\begin{aligned} &\|\varphi^2 M''_{n,s}(f)\|_1 \\ &\quad \leqslant Cn \int_0^1 \sum_{k=0}^{n-s-2} p_{n-s,k+1}(x) \int_0^1 p_{n+s,s+k+1}(t) \, |\varphi^2(t) \, f''(t)| \, dt \, dx \\ &\quad \leqslant C \int_0^1 \sum_{k=0}^{n-s-2} p_{n+s,s+k+1}(t) \, |\varphi^2(t) \, f''(t)| \, dt \leqslant C \, \|\varphi^2 f''\|_1. \end{aligned}$$

Hence, by the Riesz–Thorin theorem [2], it follows that

$$\|\varphi^2 M''_{n,s}(f)\|_p \leqslant C\,\|\varphi^2 f''\|_p, \qquad 1 \leqslant p \leqslant \infty.$$

In order to prove (7) for $i=1$, we use (4) to obtain

$$\begin{aligned}\|\varphi^2 M''_{n,s}(f)\|_{L_\infty(E_n^c)} &\leqslant \frac{C}{n}\,\|M''_{n,s}(f)\|_{L_\infty(E_n^c)}\\ &\leqslant \frac{C}{n} n^3 \left\| \sum_{k=0}^{n-s-2} p_{n-s-2,k}(x) \right.\\ &\quad \left. \times \max_{k \leqslant \lambda \leqslant k+2} \int_0^1 p_{n+s,\lambda+s}(t)\,|f(t)|\,dt \right\|_{L_\infty(E_n^c)} \leqslant Cn\,\|f\|_\infty.\end{aligned}$$

On the other hand, inside $E_n$ we have

$$|M''_{n,s}(f)(x)| \leqslant Cn \sum_{k=0}^{n-s} p''_{n-s,k}(x) \int_0^1 p_{n+s,k+s}(t)\,|f(t)|\,dt,$$

and

$$|\varphi^2(x)\,M''_{n,s}(f)(x)| \leqslant \frac{Cn}{n+s+1} \sum_{k=0}^{n-s} \varphi^2(x)\,|p''_{n-s,k}(x)|\,\|f\|_\infty.$$

Using Lemma 2 and the fact that, for $i \in N_0$ and $x \in E_n$,

$$\sum_{k=0}^{n-s} n^{i/2} \left|\frac{k}{n-s} - x\right|^i \varphi^{-i}(x)\,p_{n-s,k}(x) \leqslant C,$$

we get

$$\|\varphi^2 M''_{n,s}(f)\|_{L_\infty(E_n)} \leqslant Cn\,\|f\|_\infty.$$

Combining the estimates for $E_n^c$ and $E_n$ shows that

$$\|\varphi^2 M''_{n,s}(f)\|_\infty \leqslant Cn\,\|f\|_\infty.$$

Next we show the analogous inequality for $p=1$. Consider again $E_n^c$ first:

$$\begin{aligned}\|\varphi^2 M''_{n,s}(f)\|_{L_1(E_n^c)} &\leqslant \frac{C}{n} \int_{E_n^c} |M''_{n,s}(f)(x)|\,dx\\ &\leqslant \frac{C}{n} n^3 \int_{E_n^c} \sum_{k=0}^{n-s-2} |p_{n-s-2,k}(x)|\,|\Delta^2 a_k|\,dx.\end{aligned}$$

But

$$|\Delta^2 a_k| \leqslant 4 \max_{k \leqslant \lambda \leqslant k+2} \int_0^1 p_{n+s,\lambda+s}(t)\, |f(t)|\, dt,$$

and so

$$\begin{aligned}
&\|\varphi^2 M''_{n,s}(f)\|_{L_1(E_n^c)} \\
&\leqslant Cn^2 \int_{E_n^c} \sum_{k=0}^{n-s-2} \max_{k \leqslant \lambda \leqslant k+2} \int_0^1 p_{n+s,\lambda+2}(t)\, |f(t)|\, dt\, dx \\
&\leqslant Cn^2\, \|f\|_1 \int_{E_n^c} dx \leqslant Cn\, \|f\|_1.
\end{aligned}$$

For $x \in E_n$ we have (see the $L_\infty$ case)

$$|\varphi^2(x)\, M''_{n,s}(f)(x)| \leqslant Cn^2 \sum_{k=0}^{n-s} \sum_{i=0}^{2} n^{i/2} \left|\frac{k}{n-s} - x\right|^i \times \varphi^{-i}(x)\, p_{n-s,k}(x) \int_0^1 p_{n+s,k+s}(t)\, |f(t)|\, dt.$$

Hence, by [6, p. 129] one has

$$\begin{aligned}
\|\varphi^2 M''_{n,s}(f)\|_{L_1(E_n)} &\leqslant Cn^2 \sum_{k=0}^{n-s} \sum_{i=0}^{2} n^{i/2} \int_{E_n} \left|\frac{k}{n-s} - x\right|^i \\
&\quad \times \varphi^{-i}(x)\, p_{n-s,k}(x)\, dx \int_0^1 p_{n+s,k+s}(t)\, |f(t)|\, dt \\
&\leqslant Cn\, \|f\|_1.
\end{aligned}$$

Combining again the inequalities for $E_n^c$ and $E_n$, we obtain

$$\|\varphi^2 M''_{n,s}(f)\|_1 \leqslant Cn\, \|f\|_1,$$

and combining this with the estimate for $p = \infty$, the Riesz–Thorin theorem implies

$$\|\varphi^2 M''_{n,s}(f)\|_p \leqslant Cn\, \|f\|_p.$$

That is (7). ∎

Let

$$H_{n,m}(u) = n \left( \int_u^1 \int_0^u - \int_0^u \int_u^1 \right) (u-t)^m \times \sum_{k=0}^{n+N_1} p_{n+N_2,k+N_3}(x)\, p_{n+N_4,k+N_5}(t)\, dt\, dx,$$

with $m$, $N_i$, $i=1, 2, ..., 5$, as in (5). Since $p_{n,j}(x)=0$, $j>n$, we may suppose $N_1+N_3 \leqslant N_2$. We have

LEMMA 4. *There exists a constant* $C>0$, *such that*

$$|H_{n,m}(u)| \leqslant C\left(\frac{\varphi(u)}{\sqrt{n}}+\frac{1}{n}\right)^{m+1}.$$

*Proof.* We write $H_{n,m}(u)$ as

$$H_{n,m}(u)=n\left(\int_0^1\int_0^u-\int_0^u\int_0^1\right)dt\,dx=n\left(-\int_0^1\int_u^1+\int_u^1\int_0^1\right)dt\,dx.$$

Using the fact that $(n+1)\int_0^1 p_{n,k}(x)\,dx=1$ and changing the order of integration, then integrating by parts and by (2), we have

$$\begin{aligned}H_{n,m}(u)=n\frac{n+N_4}{m+1}&\left\{\int_u^1 dx\int_0^1 (u-t)^{m+1}\right.\\ &\times\sum_{k=-N_5+1}^{n+N_1} p_{n+N_2,k+N_3}(x)\{p_{n-N_4-1,k+N_5-1}(t)\\ &-p_{n+N_4-1,k+N_5}(t)\}-\frac{1}{n+N_2+1}\\ &\times\sum_{k=0}^{n+N_1}\int_u^1 (u-t)^{m+1}\{p_{n+N_4-1,k+N_5-1}(t)\\ &\left.-p_{n+N_4-1,k+N_5}(t)\}\,dt\right\}+n\int_u^1 dx\int_0^1(u-t)^m\\ &\times\sum_{k=0}^{-N_5} p_{n+N_2,k+N_3}(x)\,p_{n+N_4,k+N_5}(t)\,dt. \end{aligned}\tag{9}$$

But $p_{n,j}(x)=0$, if $j<0$. So if $N_5 \leqslant 0$

$$\begin{aligned}&n\int_u^1 dx\int_0^1 (u-t)^m\sum_{k=0}^{-N_5} p_{n+N_2,k+N_3}(x)\,p_{n+N_4,k+N_5}(t)\,dt\\ &\quad=n\int_u^1 dx\int_0^1 (u-t)^m\,p_{n+N_2,N_3-N_5}(x)\,p_{n+N_4,0}(t)\,dt.\end{aligned}$$

Thus, if $N_3 - N_5 < 0$, the last integral equals zero. If $0 \leqslant N_3 - N_5$, then, since

$$\left| \int_0^1 |u-t|^m \, p_{n+N_4,0}(t)\, dt \right| \leqslant C \left\{ \frac{u^m}{n} + \int_0^1 t^m (1-t)^{n+N_4}\, dt \right\}$$

$$\leqslant C \left( \frac{u^m}{n} + \frac{1}{n^{m+1}} \right),$$

$$\int_u^1 p_{n+N_2, N_3-N_5}(x)\, dx \leqslant C \left\{ n^{N_3-N_5} \sum_{i=0}^{N_3-N_5} n^{-i-1} u^{N_3-N_5-i} \times (1-u)^{n+N_2-N_3+N_5+i+1} \right\},$$

and for fixed $N$, $u^N(1-u)^{n-N} \leqslant Cn^{-N}$, we obtain

$$\left| n \int_u^1 dx \int_0^1 (u-t)^m \sum_{k=0}^{-N_5} p_{n+N_2, k+N_3}(x)\, p_{n+N_4, k+N_5}(t)\, dt \right|$$

$$\leqslant C \left( \frac{\varphi(u)}{\sqrt{n}} + \frac{1}{n} \right)^{m+1}.$$

If $N_5 > 0$, by the same reasoning as above, we have also the same estimate. For the second part of (9), we have

$$\sum_{k=0}^{n+N_1} \int_u^1 (u-t)^{m+1} \{ p_{n+N_4-1, k+N_5-1}(t) - p_{n+N_4-1, k+N_5}(t) \}\, dt$$

$$= -\int_u^1 (u-t)^{m+1}\, p_{n+N_4-1, n+N_1+N_5}(t)\, dt + O\left( \frac{1}{n^{m+2}} \right).$$

On the other hand, using the Abel transformation and (2) for the first part of (9), it is not difficult to obtain that

$$\sum_{k=-N_5+1}^{n+N_1} p_{n+N_2, k+N_3}(x) \{ p_{n+N_4-1, k+N_5-1}(t) - p_{n+N_4-1, k+N_5}(t) \}$$

$$= -p_{n+N_2, n+N_1+N_3}(x)\, p_{n+N_4-1, n+N_1+N_5}(t)$$

$$+ p_{n+N_2, N_3-N_5}(x)\, p_{n+N_4-1, 0}(t)$$

$$- \frac{1}{n+N_2+1} \sum_{k=-N_5}^{n+N_1-1} p_{n+N_4-1, k+N_5}(t)\, p_{n+N_2+1, k+N_3+1}(x).$$

Thus it follows that

$$H_{n,m}(u)=\frac{n(n+N_4)}{m+1}\left\{\frac{-1}{n+N_2+1}\int_u^1 dx\int_0^1 (u-t)^{m+1}\right.$$
$$\times\sum_{k=-N_5}^{n+N_1-1} p'_{n+N_2+1,k+N_3+1}(x)\, p_{n+N_4-1,k+N_5}(t)\, dt$$
$$-\int_u^1 dx\int_0^1 (u-t)^{m+1}\, p_{n+N_2,n+N_1+N_3}(x)\, p_{n+N_4-1,n+N_1+N_5}(t)\, dt$$
$$\left.+\frac{1}{n+N_2+1}\int_u^1 (u-t)^{m+1}\, p_{n+N_4-1,n+N_1+N_5}(t)\, dt\right\}$$
$$+O\left(\left(\frac{\varphi(u)}{\sqrt{n}}+\frac{1}{n}\right)^{m+1}\right).$$

Direct computation shows that

$$\left|\int_u^1 dx\int_0^1 (u-t)^{m+1}\, p_{n+N_2,n+N_1+N_3}(x)\, p_{n+N_4-1,n+N_1+N_5}(t)\, dt\right.$$
$$\left.-\frac{1}{n+N_2+1}\int_u^1 (u-t)^{m+1}\, p_{n+N_4-1,n+N_1+N_5}(t)\, dt\right|\leqslant\frac{C}{n^{m+3}}.$$

Therefore, since $N_2+1>N_1+N_3$,

$$H_{n,m}(u)=-\frac{n(n+N_4)}{(m+1)(n+N_2+1)}\int_u^1 dx\int_0^1 (u-t)^{m+1}$$
$$\times\sum_{k=-N_5}^{n+N_1-1} p'_{n+N_2+1,k+N_3+1}(x)\, p_{n+N_4-1,k+N_5}(t)\, dt$$
$$+O\left(\left(\frac{\varphi(u)}{\sqrt{n}}+\frac{1}{n}\right)^{m+1}\right)$$
$$=\frac{n(n+N_4)}{(n+N_2+1)(m+1)}\sum_{k=0}^{n+N_1+N_5-1} p_{n+N_2+1,k-N_5+N_3+1}(u)$$
$$\times\int_0^1 (u-t)^{m+1}\, p_{n+N_4-1,k}(t)\, dt$$
$$+O\left(\left(\frac{\varphi(u)}{\sqrt{n}}+\frac{1}{n}\right)^{m+1}\right).$$

Now, using (5), we get the assertion of this lemma. ∎

Defining

$$M(v) = n \sum_{k=0}^{n-s} p_{n-s,k}(v) \int_0^1 p_{n+s,k+s}(t) \int_x^t (t-u)^3 f^{(4)}(u)\, du\, dt,$$

we prove

LEMMA 5. *For* $1 \leqslant p \leqslant \infty$ *there holds*

$$\|M(x)\|_p + \frac{1}{n} \left\| \frac{d}{dv} M(v) \bigg|_{v=x} \right\|_p + \frac{1}{n} \left\| \varphi^2(x) \frac{d^2}{dv^2} M(v) \bigg|_{v=x} \right\|_p$$

$$\leqslant C \left\{ \frac{1}{n^2} \|\varphi^4 f^{(4)}\|_p + \frac{1}{n^4} \|f^{(4)}\|_p \right\}.$$

*Proof.* From [6, p. 141]) it is known that

$$\left| \int_x^t (t-u)^3 f^{(4)}(u)\, du \right| \leqslant \begin{cases} (t-x)^4 \varphi^{-4}(x) M(\varphi^4 f^{(4)})(x), & x \in E_n \\ (t-x)^4 M(f^{(4)})(x), & x \in E_n^c. \end{cases}$$

Here $M(F)(x)$ is the Hardy–Littlewood function of $F$. Thus we have the inequality of this lemma for $1 < p \leqslant \infty$, using the Hardy–Littlewood inequality, Lemma 2 for $x \in E_n$, and the fact that $|\varphi(x)| \leqslant C/n$ for $x \in E_n^c$ and (5).

If $p = 1$, then

$$\|M(x)\|_1 \leqslant n \left\| \sum_{k=0}^{n-s} p_{n-s,k}(x) \int_0^1 p_{n+s,k+s}(t) \int_x^t (t-u)^3 |f^{(4)}(u)|\, du\, dt \right\|_1$$

$$= n \int_0^1 |f^{(4)}(u)| \left\{ \int_0^1 \int_0^u - \int_0^u \int_0^1 \right\}$$

$$\times (u-t)^3 \sum_{k=0}^{n-s} p_{n-s,k}(x)\, p_{n+s,k+s}(t)\, dt\, dx\, du.$$

By (2),

$$\frac{1}{n} \left| \frac{d}{dv} M(v) \bigg|_{v=x} \right| \leqslant Cn \sum_{k=0}^{n-s} p_{n-s-1,k}(x) \int_0^1 \{ p_{n+s,k+s}(t)$$

$$+ p_{n+s,k+s-1}(t) \} \int_x^t (t-u)^3 |f^{(4)}(u)|\, du\, dt.$$

We thus get

$$\frac{1}{n}\left\|\frac{d}{dv}M(v)\bigg|_{v=x}\right\|_1$$

$$\leqslant Cn\left\{\int_0^1 |f^{(4)}(u)| \left\{\int_0^1\int_0^u - \int_0^u\int_0^1\right\}(u-t)^3\right.$$

$$\times\sum_{k=0}^{n-s} p_{n-s-1,k}(x)\, p_{n+s,k+s}(t)\, dt\, dx\, du$$

$$+\int_0^1 |f^{(4)}(u)| \left\{\int_0^1\int_0^u - \int_0^u\int_0^1\right\}(u-t)^3$$

$$\left.\times\sum_{k=0}^{n-s} p_{n-s-1,k}(x)\, p_{n+s,k+s-1}(t)\, dt\, dx\, du\right\}.$$

Using (3), we obtain

$$\frac{\varphi^2(x)}{n}\left|\frac{d^2}{dv^2}M(v)\bigg|_{v=x}\right|$$

$$\leqslant C\sum_{k=0}^{n-s-2} p_{n-s,k+1}(x)\int_0^1 p_{n+s,k+s+1}(t)\,\varphi^2(t)\int_x^t (t-u)\,|f^{(4)}(u)|\,du\,dt.$$

So, like in [10],

$$\frac{1}{n}\left\|\varphi(x)\frac{d^2}{dv^2}M(v)\bigg|_{v=x}\right\|_{L[0,1/2]}$$

$$\leqslant C\left\{\int_0^1 |f^{(4)}(u)|\,\varphi^2(u)\left\{\int_u^1\int_0^u - \int_0^u\int_u^1\right\}(u-t)\right.$$

$$\times\sum_{k=0}^{n-s-2} p_{n-s,k+1}(x)\, p_{n+s,k+s+1}(t)\, dt\, dx\, du$$

$$+\int_0^1 |f^{(4)}(u)|\,\varphi^4(u)\left\{\int_u^1\int_0^u - \int_0^u\int_u^1\right\}(u-t)$$

$$\left.\times\sum_{k=0}^{n-s-2} p_{n-s-2,k}(x)\, p_{n+s+2,k+s+2}(t)\, dt\, dx\, du\right\}.$$

The same estimate holds also for $x\in[1/2, 1]$. Now, using Lemma 4, we obtain the claim of Lemma 5. ▌

We introduce the auxiliary operator $L_n$ defined by

$$L_n(f) = C_n M_{n,s}(f) - \frac{C_n}{n-s}\{\varphi^2 M''_{n,s}(f) + (s+1)(\varphi^2)'\, M'_{n,s}(f)\},$$

where

$$C_n = \frac{n+s+1}{(n+1)\,\alpha(n,s)}.$$

This operator $L_n$ satisfies an inequality as given in

LEMMA 6. *For* $1 \leqslant p \leqslant \infty$,

$$\|L_n(f)-f\|_p \leqslant C\left\{\omega_\varphi^3\left(f, \frac{1}{\sqrt{n}}\right)_p + \frac{1}{n}\|f\|_p\right\}.$$

*Proof.* Note first that

$$L_n(1)=1, \qquad L_n(t;x)=x, \qquad L_n(t^2;x)=x^2+O\left(\frac{1}{n^2}\right).$$

By Lemma 3, $L_n$ is a bounded linear operator from $L_p(I)$ into $L_p(I)$. Since for a polynomial $P$ with degree not larger than $[\sqrt{n}]$, $L_n(P)$ is also a polynomial of the same degree, by [6, p. 91] one has

$$\|L_n(P)-P\|_p \leqslant C\,\|L_n(P)-P\|_{L_p(E_n)}.$$

On the other hand, using Taylor's formula, we get

$$\|L_n(P)-P\|_p \leqslant C\left\{\|P''\|_{L_p(E_n)}\frac{1}{n^2} + \|L_n((t-\cdot)^3)\,P^{(3)}\|_{L_p(E_n)} + \|L_n(R)\|_{L_p(E_n)}\right\}.$$

Here,

$$L_n(R;x) = L_n\left(\int_x^t (t-u)^3\,P^{(4)}(u)\,du;x\right).$$

It is easy to calculate that (see [9])

$$|L_n((t-x)^3;x)| \leqslant C\left\{|M_{n,s}((t-x)^3;x)| + \frac{\varphi^2(x)}{n}\,|M''_{n,s}((t-x)^3;u)|_{u=x}| + \frac{1}{n}\,|M'_{n,s}((t-x)^3;u)|_{u=x}|\right\} \leqslant C\left(\frac{\varphi^2(x)}{n^2}+\frac{1}{n^4}\right).$$

Thus, since $x \in E_n$, we have (see [6, p. 135])

$$\|L_n((t-\cdot)^3)\,P^{(3)}\|_{L_p(E_n)} \leqslant \left\|\frac{\varphi^2}{n^2}\,P^{(3)}\right\|_p \leqslant \frac{C}{n^2}\left(\|\varphi^4 P^{(4)}\|_p + \|P\|_p\right).$$

By Lemma 5 we obtain, because $P$ is a polynomial of degree not larger than $[\sqrt{n}]$, and $C/n \leqslant \varphi^2(x)$ for $x \in E_n$,

$$\|L_n(R)\|_p \leqslant C\left\{\frac{1}{n^2}\|P^{(4)}\varphi^4\|_p + \frac{1}{n^4}\|P^{(4)}\|_p\right\}$$

$$\leqslant C\left\{\frac{1}{n^2}\|P^{(4)}\varphi^4\|_p + \frac{1}{n^4}\|P^{(4)}\|_{L_p(E_n)}\right\}$$

$$\leqslant C\left\{\frac{1}{n^2}\|\varphi^4 P^{(4)}\|_p\right\}.$$

In order to estimate $\|P''\|_{L_p(E_n)}$, observe that, if $1 \leqslant p < \infty$, then (see [6, p. 135])

$$\|P''\|_p \leqslant C\{\|\varphi^4 P^{(4)}\|_p + \|P\|_p\}.$$

If $p = \infty$, we write

$$\left|P''(x) - P''\left(\frac{1}{2}\right)\right| \leqslant \left|\int_x^{1/2} P^{(3)}(u)\, du\right| \leqslant \frac{C}{\varphi(x)}\|\varphi^3 P^{(3)}\|_\infty,$$

and use the estimate

$$|P''(\tfrac{1}{2})| \leqslant Cn\,\|P\|_\infty,$$

which is valid for $P \in \Pi_{[\sqrt{n}]}$. Therefore,

$$\frac{1}{n^2}\|P''\|_{L_\infty(E_n)} \leqslant \frac{C}{n^2}\{n\,\|P\|_\infty + n^{1/2}\,\|\varphi^3 P^{(3)}\|_\infty\}$$

$$\leqslant C\left\{\frac{1}{n}\|P\|_\infty + \frac{1}{n^{3/2}}\|\varphi^3 P^{(3)}\|_\infty\right\}.$$

Hence, for $1 \leqslant p < \infty$,

$$\|L_n(P) - P\|_p \leqslant C\left\{\frac{1}{n^2}\|P\|_p + \frac{1}{n^2}\|\varphi^4 P^{(4)}\|_p\right\},$$

and for $p = \infty$,

$$\|L_n(P) - P\|_\infty \leqslant C\left\{\frac{1}{n}\|P\|_\infty + \frac{1}{n^{3/2}}\|\varphi^3 P^{(3)}\|_\infty + \frac{1}{n^2}\|\varphi^4 P^{(4)}\|_\infty\right\}.$$

Now, let $P$ satisfy (see [6, p. 84])

$$\|P - f\|_p \leqslant CE_{[\sqrt{n}]}(f) \leqslant C\omega_\varphi^3\left(f, \frac{1}{\sqrt{n}}\right)_p,$$

and

$$n^{-i/2}\,\|\varphi^i P^{(i)}\|_p \leqslant C\omega_\varphi^3\left(f, \frac{1}{\sqrt{n}}\right), \qquad i=3,4.$$

Thus

$$\|L_n(f)-f\|_p \leqslant C\left\{\omega_\varphi^3\left(f, \frac{1}{\sqrt{n}}\right)_p + \frac{1}{n}\|f\|_p\right\}. \quad \blacksquare$$

## 3. Main Result

We are now ready to prove the main result of this paper. It will be formulated in terms of a function $\omega$ having properties similar to those of a second order modulus of continuity, namely that $\omega$ is increasing such that

$$0 \leqslant \omega(kt) \leqslant Ak^2\omega(t) \qquad \text{for} \quad t>0 \text{ and } k \in N.$$

The theorem below supplements and extends the direct theorem given by Heilmann and Müller [9] as indicated in the introduction.

THEOREM. *Let* $1 \leqslant s$ *and* $f \in L_p^s(I)$, $1 \leqslant p < \infty$. *Then*

$$\|(M_n(f)-f)^{(s)}\|_p \leqslant C\left\{\omega\left(\frac{1}{\sqrt{n}}\right)+\frac{1}{n}\right\}$$

*if and only if*

$$\omega_\varphi^2(f^{(s)}, t)_p \leqslant C\{\omega(t)+t^2\},$$

*where* $C=C_f$ *is independent of* $t$ *(and* $n$*).*

*Proof.* ($\Leftarrow$) It was shown in [9, Theorem 3.1] that, for $1 \leqslant p < \infty$,

$$\begin{aligned}\|(M_n(f)-f)^{(s)}\|_p &\leqslant C\left\{\omega_\varphi^2\left(f^{(s)}, \frac{1}{\sqrt{n}}\right)+\frac{1}{n}\|f\|_p\right\}\\ &\leqslant C\left\{\omega\left(\frac{1}{\sqrt{n}}\right)+\frac{1}{n}\right\}.\end{aligned}$$

($\Rightarrow$) The only case of interest is the one in which $Ct^2 \leqslant \omega(t)$. We assume that

$$\|(M_n(f)-f)^{(s)}\|_p \leqslant C\omega\left(\frac{1}{\sqrt{n}}\right).$$

Thus, since $|1-C_n|<C/n$,

$$\|C_n M_{n,s}(f^{(s)})-f^{(s)}\|_p \leqslant C_n \|(M_n(f)-f)^{(s)}\|_p + |1-C_n| \, \|f^{(s)}\|_p$$
$$\leqslant C\omega\left(\frac{1}{\sqrt{n}}\right). \tag{10}$$

Furthermore,

$$\omega_\varphi^4(f^{(s)}, t)_p \leqslant C\{\|f^{(s)}-M_{n,s}(f^{(s)})\|_p + t^4 \|\varphi^4(M_{n,s}(f^{(s)}))^{(4)}\|_p\}.$$

Lemma 3 shows that, for any $g^{(s)} \in L_p^4(I)$,

$$\omega_\varphi^4(f^{(s)}, t)_p \leqslant C\{\|f^{(s)}-M_{n,s}(f^{(s)})\|_p + t^4(n^2 \|(f-g)^{(s)}\|_p + \|\varphi^4 g^{(s+4)}\|_p\}.$$

Taking the inf over $g^{(s)} \in L_p^4(I)$ and using the equivalence of $K$-functionals and corresponding moduli of smoothness (see [6, p. 11]), by the above condition on $\omega$, we get

$$\omega_\varphi^4(f^{(s)}, t)_p \leqslant C\left\{\omega\left(\frac{1}{\sqrt{n}}\right) + t^4 n^2 \omega_\varphi^4\left(f^{(s)}, \frac{1}{\sqrt{n}}\right)_p\right\}.$$

Now, Lemma 1 implies

$$\omega_\varphi^4(f^{(s)}, t)_p \leqslant C\omega(t).$$

Using Marchaud's inequality and by the condition on $\omega$ this implies

$$\omega_\varphi^3(f^{(s)}, t)_p \leqslant C\omega(t).$$

Combining this with (10) it follows from Lemma 6 that

$$\|\varphi^2 M''_{n,s}(f^{(s)}) + (s+1)(\varphi^2)' \, M'_{n,s}(f^{(s)})\|_p \leqslant Cn\omega\left(\frac{1}{\sqrt{n}}\right). \tag{11}$$

If we can prove that

$$\|M'_{n,s}(f^{(s)})\|_p \leqslant Cn\omega\left(\frac{1}{\sqrt{n}}\right), \tag{12}$$

then,

$$\|M_{n,s}(f^{(s)})-f^{(s)}\|_p + \frac{1}{n} \|\varphi^2 M''_{n,s}(f^{(s)})\|_p \leqslant C\omega\left(\frac{1}{\sqrt{n}}\right).$$

Thus, using the equivalence of the $K$-functional and the modulus of smoothness (see [6, p. 11]), we obtain

$$\omega_\varphi^2(f^{(s)}, t)_p \leqslant C\omega(t).$$

Hence, we only have to prove (12). In order to do this, we write $g = M_{n,s}(f^{(s)})$ and

$$h(x) = \varphi^2 g'' + (s+1)(\varphi^2)' g' = (\varphi^2)^{-s} (g'(\varphi^2)^{s+1})'.$$

(11) shows $\|h\|_p \leqslant Cn\omega(1/\sqrt{n})$. Furthermore,

$$g'(x) = \frac{1}{\varphi^{2s+2}} \int_0^x h(t)\, \varphi^{2s}(t)\, dt.$$

So, by Hardy's inequality, we have for $1 < p < \infty$

$$\|g'\|_{L_p[0,1/2]} \leqslant \left\| \frac{\varphi^{2s}(x)}{\varphi^{2s+2}(x)} \int_0^x |h(t)|\, dt \right\|_{L_p[0,1/2]} \leqslant C\, \|h\|_p.$$

For $p = 1$, by changing the order of integration, one gets

$$\begin{aligned} \|g'\|_{L[0,1/2]} &\leqslant \int_0^{1/2} \frac{1}{\varphi^{2s+2}(x)} \int_0^x |h(t)|\, \varphi^{2s}(t)\, dt\, dx \\ &\leqslant 2 \int_0^{1/2} |h(t)| \int_t^{1/2} \frac{t^s}{x^{s+1}}\, dx\, dt. \end{aligned}$$

Since $1 \leqslant s$, we obtain

$$\|g'\|_{L[0,1/2]} \leqslant C\, \|h\|_1.$$

In the same way one can show that

$$\|g'\|_{L_p[1/2,1]} \leqslant C\, \|h\|_p.$$

Thus, for all $1 \leqslant p < \infty$,

$$\|g'\|_p \leqslant C\, \|h\|_p.$$

That implies (12) and the proof is complete. ∎

*Remark.* In case $p = \infty$, we are able to prove the direct theorem (e.g., using Lemma 3 and Lemma 6)

$$\|(M_n(f) - f)^{(s)}\|_\infty \leqslant C \left\{ \omega_\varphi^2 \left( f^{(s)}, \frac{1}{\sqrt{n}} \right)_\infty + \omega \left( f^{(s)}, \frac{1}{n} \right)_\infty + \frac{1}{n} \|f^{(s)}\|_\infty \right\}.$$

Thus, using the same method as in the proof of the theorem, we have

$$\|(M_n(f)-f)^{(s)}\|_\infty \leqslant C\left\{\omega\left(\frac{1}{\sqrt{n}}\right)+\frac{1}{n}\right\}$$

if and only if

$$\omega_\varphi^2(f^{(s)}, t)_\infty+\omega(f^{(s)}, t^2)_\infty \leqslant C(\omega(t)+t^2).$$

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